
2020 考研数学二真题及答案解析

一、选择题：1~8 小题，每小题 4 分，共 32 分.下列每题给出的四个选项中，只有一个选项是符合题目要求的，请将选项前的字母填在答题纸指定位置上.

1. $x \rightarrow 0^+$ ，下列无穷小量中最高阶是 ()

A. $\int_0^x (e^{t^2} - 1) dt$

B. $\int_0^x \ln(1 + \sqrt{t^3}) dt$

C. $\int_0^{\sin x} \sin t^2 dt$

D. $\int_0^{1-\cos x} \sqrt{\sin^3 t} dt$

答案：D

解析：A. $\int_0^x (e^{t^2} - 1) dt \sim \int_0^x t^2 dt = \frac{x^3}{3}$

B. $\int_0^x \ln(1 + \sqrt{t^3}) dt \sim \int_0^x t^{\frac{3}{2}} dt = \frac{2}{5} x^{\frac{5}{2}}$

C. $\int_0^{\sin x} \sin t^2 dt \sim \int_0^x t^2 dt = \frac{1}{3} x^3$

D. $\int_0^{1-\cos x} \sqrt{\sin^3 t} dt \sim \int_0^{\frac{1}{2}x^2} t^{\frac{3}{2}} dt$
 $= \frac{2}{5} t^{\frac{5}{2}} \Big|_0^{\frac{1}{2}x^2}$
 $= \frac{2}{5} \left(\frac{1}{2} x^2 \right)^{\frac{5}{2}} = \frac{1}{10\sqrt{2}} x^5$

2. $f(x) = \frac{e^{x-1} \ln|1+x|}{(e^x - 1)(x-2)}$ 第二类间断点个数 ()

A.1

B.2

C.3

D.4

答案：C

解析： $x=0, x=2, x=1, x=-1$ 为间断点

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \lim_{x \rightarrow 0} \frac{e^{-1} \ln |1+x|}{-2x} = -\frac{e^{-1}}{2} \lim_{x \rightarrow 0} \frac{\ln |x+1|}{x} = -\frac{e^{-1}}{2}$$

$x = 0$ 为可去间断点

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \infty$$

$x = 2$ 为第二类间断点

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \infty$$

$x = 1$ 为第二类间断点

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \infty$$

$x = -1$ 为第二类间断点

$$3. \int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx =$$

A. $\frac{\pi^2}{4}$

B. $\frac{\pi^2}{8}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{8}$

答案: A

解析:

$$\int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx$$

令 $u = \sqrt{x}$, 则

$$\text{原式} = \int_0^1 \frac{\arcsin u}{\sqrt{u^2(1-u^2)}} \cdot 2u du$$

$$= 2 \int_0^1 \frac{\arcsin u}{\sqrt{1-u^2}} du$$

$$\text{令 } u = \sin t \Rightarrow \int_0^{\frac{\pi}{2}} \frac{t}{\cos t} \cos t dt$$

$$= 2 \cdot \frac{1}{2} t^2 \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$$

4. $f(x) = x^2 \ln(1-x)$, $n \geq 3$ 时, $f^{(n)}(0) =$

A. $-\frac{n!}{n-2}$

B. $\frac{n!}{n-2}$

C. $-\frac{(n-2)!}{n}$

D. $\frac{(n-2)!}{n}$

答案: A

解析:

$$f(x) = x^2 \ln(1-x), n \geq 3$$

$$f^{(n)}(x) = C_n^0 x^2 [\ln(1-x)]^{(n)} + C_n^1 (x^2)' [\ln(1-x)]^{(n-1)} + C_n^2 (x^2)'' [\ln(1-x)]^{(n-2)}$$

$$\because [\ln(1-x)]^{(n)} = \frac{(n-1)!(-1)^n}{(1-x)^n}$$

$$[\ln(1-x)]^{(n-1)} = \frac{(n-2)!(-1)^{n-1}}{(1-x)^{n-1}}$$

$$[\ln(1-x)]^{(n-2)} = \frac{(n-3)!(-1)^{n-2}}{(1-x)^{n-2}}$$

$$(x^2)' = 2x; (x^2)'' = 2.$$

$$\therefore f^{(n)}(x) = x^2 \cdot \frac{(n-1)!(-1)^n}{(1-x)^n} + 2n \cdot x \cdot \frac{(n-2)!(-1)^{n-1}}{(1-x)^{n-1}} + 2 \cdot \frac{n \cdot (n-1)}{2} \cdot \frac{(n-3)!(-1)^{n-2}}{(1-x)^{n-2}}$$

$$\therefore f^{(n)}(0) = -\frac{n!}{n-2}.$$

5. 关于函数 $f(x, y) = \begin{cases} xy & xy \neq 0 \\ x & y = 0 \\ y & x = 0 \end{cases}$ 给出以下结论

① $\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 1$

$$\textcircled{2} \left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} = 1$$

$$\textcircled{3} \lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$$

$$\textcircled{4} \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y) = 0 \text{ 正确的个数是}$$

A.4

B.3

C.2

D.1

答案: B

解析:

$$\textcircled{1} \left. \frac{\partial f}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{x-0}{x} = 1$$

$$\textcircled{2} xy \neq 0 \text{ 时, } \frac{\partial f}{\partial x} = y$$

$$y = 0 \text{ 时, } \frac{\partial f}{\partial x} = 1$$

$$x = 0 \text{ 时, } \frac{\partial f}{\partial x} = 0$$

$$\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{f'_x(0,y) - f'_x(0,0)}{y} = \lim_{y \rightarrow 0} \frac{-1}{y} \text{ 不存在.}$$

$$\textcircled{3} xy \neq 0, \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} xy = 0$$

$$y = 0, \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} x = 0$$

$$x = 0, \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} y = 0$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$$

$$\textcircled{4} xy \neq 0, \lim_{x \rightarrow 0} f(x,y) = \lim_{x \rightarrow 0} xy = 0$$

$$y = 0, \lim_{x \rightarrow 0} f(x,y) = \lim_{x \rightarrow 0} x = 0$$

$$x = 0, \lim_{x \rightarrow 0} f(x,y) = \lim_{x \rightarrow 0} y = y$$

$$\text{从而 } \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y) = 0.$$

6. 设函数 $f(x)$ 在区间 $[-2, 2]$ 上可导, 且 $f'(x) > f(x) > 0$, 则 ()

A. $\frac{f(-2)}{f(-1)} > 1$

B. $\frac{f(0)}{f(-1)} > e$

C. $\frac{f(1)}{f(-1)} < e^2$

D. $\frac{f(2)}{f(-1)} < e^3$

答案: B

解析: 由 $f'(x) > f(x) > 0$ 知

$$\frac{f'(x)}{f(x)} - 1 > 0$$

即 $(\ln f(x) - x)' > 0$

令 $F(x) = \ln f(x) - x$, 则 $F(x)$ 在 $[-2, 2]$ 上单增

因 $-2 < -1$, 所以 $F(-2) < F(-1)$

即 $\ln f(-2) + 2 < \ln f(-1) + 1$

$$\frac{f(-1)}{f(-2)} > e$$

同理, $-1 < 0, F(-1) < F(0)$

即 $\ln f(-1) + 1 < \ln f(0)$

$$\frac{f(0)}{f(-1)} > e$$

7. 设四阶矩阵 $A = (a_{ij})$ 不可逆, a_{12} 的代数余子式 $A_{12} \neq 0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ 为矩阵 A 的列向量

组. A^* 为 A 的伴随矩阵. 则方程组 $A^*x = \mathbf{0}$ 的通解为 ().

A. $x = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$, 其中 k_1, k_2, k_3 为任意常数

B. $x = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_4$, 其中 k_1, k_2, k_3 为任意常数

C. $x = k_1\alpha_1 + k_2\alpha_3 + k_3\alpha_4$, 其中 k_1, k_2, k_3 为任意常数.

D. $x = k_1\alpha_2 + k_2\alpha_3 + k_3\alpha_4$, 其中 k_1, k_2, k_3 为任意常数

答案: C

解析:

$\because A$ 不可逆

$$\therefore |A|=0$$

$$\therefore A_{12} \neq 0 \quad \therefore r(A)=3$$

$$\therefore r(A^*)=1$$

$\therefore A^*x=0$ 的基础解系有 3 个线性无关的解向量.

$$\therefore A^*A=|A|E=0$$

$\therefore A$ 的每一列都是 $A^*x=0$ 的解

$$\text{又} \therefore A_{12} \neq 0 \quad \therefore \alpha_1, \alpha_3, \alpha_4 \text{ 线性无关}$$

$$\therefore A^*x=0 \text{ 的通解为 } x=k_1\alpha_1+k_2\alpha_3+k_3\alpha_4$$

8. 设 A 为 3 阶矩阵, α_1, α_2 为 A 属于特征值 1 的线性无关的特征向量, α_3 为 A 的属于特征

值 -1 的特征向量, 则满足 $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 的可逆矩阵 P 可为 ().

A. $(\alpha_1 + \alpha_3, \alpha_2, -\alpha_3)$

B. $(\alpha_1 + \alpha_2, \alpha_2, -\alpha_3)$

C. $(\alpha_1 + \alpha_3, -\alpha_3, -\alpha_3)$

D. $(\alpha_1 + \alpha_2, -\alpha_3, -\alpha_2)$

答案: D

解析:

$$A\alpha_1 = \alpha_1, A\alpha_2 = \alpha_2$$

$$A\alpha_3 = -\alpha_3$$

$$\therefore P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore P$ 的 1, 3 两列为 1 的线性无关的特征向量 $\alpha_1 + \alpha_2, \alpha_2$

P 的第 2 列为 A 的属于 -1 的特征向量 α_3 .

$$\therefore P = (\alpha_1 + \alpha_2, -\alpha_3, \alpha_2)$$

二、填空题：9~14 小题，每小题 4 分，共 24 分. 请将答案写在答题纸指定位置上.

$$9. \text{ 设 } \begin{cases} x = \sqrt{t^2 + 1} \\ y = \ln(t + \sqrt{t^2 + 1}) \end{cases}, \text{ 则 } \frac{d^2 y}{dx^2} \Big|_{t=1} = \underline{\hspace{2cm}}.$$

解析：

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t + \sqrt{t^2 + 1}} \left(1 + \frac{t}{\sqrt{t^2 + 1}} \right)}{\frac{t}{\sqrt{t^2 + 1}}}$$

$$= \frac{1}{t}$$

$$\frac{dy^2}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d\left(\frac{dy}{dt}\right)}{dt}}{\frac{dx}{dt}} = \frac{-\frac{1}{t^2}}{\frac{t}{\sqrt{t^2 + 1}}}$$

$$= -\frac{\sqrt{t^2 + 1}}{t^3}$$

$$\frac{dy^2}{dx^2} \Big|_{t=1} = -\sqrt{2}$$

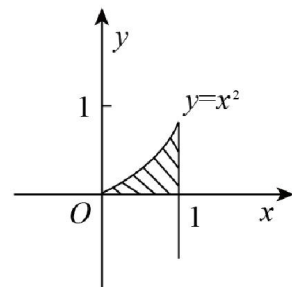
$$10. \int_0^1 dy \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} dx = \underline{\hspace{2cm}}.$$

$$\text{解析：} \int_0^1 dy \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} dx$$

$$= \int_0^1 dx \int_0^{x^2} \sqrt{x^3 + 1} dy$$

$$= \int_0^1 \sqrt{x^3 + 1} dx \int_0^{x^2} dy$$

$$= \int_0^1 \sqrt{x^3 + 1} x^2 dx$$



$$\begin{aligned}
 &= \frac{1}{3} \int_0^1 (x^3 + 1)^{\frac{1}{2}} d(x^3 + 1) \\
 &= \frac{1}{3} \cdot \frac{2}{3} (x^3 + 1)^{\frac{3}{2}} \Big|_0^1 \\
 &= \frac{2}{9} \left(2^{\frac{3}{2}} - 1 \right)
 \end{aligned}$$

11. 设 $z = \arctan[xy + \sin(x + y)]$, 则 $dz|_{(0,\pi)} = \underline{\hspace{2cm}}$.

解析:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + [xy + \sin(x + y)]^2} [y + \cos(x + y)], \quad \frac{\partial z}{\partial x} \Big|_{(0,\pi)} = \pi - 1$$

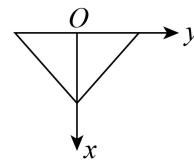
$$\frac{\partial z}{\partial y} = \frac{1}{1 + [xy + \sin(x + y)]^2} [x + \cos(x + y)], \quad \frac{\partial z}{\partial y} \Big|_{(0,\pi)} = -1$$

$$\therefore \frac{\partial z}{\partial x} \Big|_{(0,\pi)} = (\pi - 1)dx - dy$$

12. 斜边长为 $2a$ 等腰直角三角形平板铅直地沉没在水中, 且斜边与水面相齐, 设重力加速度为 g , 水密度为 ρ , 则该平板一侧所受的水压力为

解析: 建立直角坐标系, 如图所示

$$\begin{aligned}
 F &= 2 \int_0^a \rho g x \cdot (a - x) dx \\
 &= 2 \rho g \int_0^a ax - x^2 dx \\
 &= 2 \rho g \left(\frac{a}{2} x^2 - \frac{1}{3} x^3 \right) \Big|_0^a \\
 &= \frac{1}{3} \rho g a^3
 \end{aligned}$$



13. 设 $y = y(x)$ 满足 $y'' + 2y' + y = 0$, 且 $y(0) = 0, y'(0) = 1$, 则 $\int_0^{+\infty} y(x) dx = \underline{\hspace{2cm}}$

解析: 特征方程 $\lambda^2 + 2\lambda + 1 = 0$

$$\therefore \lambda_1 = \lambda_2 = -1$$

$$\therefore y(x) = (C_1 + C_2 x) e^{-x}$$

$$\begin{aligned} \int_0^{+\infty} y(x) dx &= -\int_0^{+\infty} y''(x) + 2y'(x) dx \\ &= -[y'(x) + 2y(x)] \Big|_0^{+\infty} \\ &= [y'(0) + 2y(0)] = 1 \end{aligned}$$

14. 行列式 $\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = \underline{\hspace{2cm}}$

解析:

$$\begin{aligned} &\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = \begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix} \\ &= \begin{vmatrix} 0 & a & -1+a^2 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix} = - \begin{vmatrix} a & -1+a^2 & 1 \\ a & 1 & -1 \\ 0 & a & a \end{vmatrix} \\ &= - \begin{vmatrix} a & a^2-2 & 1 \\ a & 2 & -1 \\ 0 & 0 & a \end{vmatrix} = a^4 - 4a^2. \end{aligned}$$

三、解答题: 15~23 小题, 共 94 分. 请将解答写在答题纸指定位置上. 解答写出文字说明、证明过程或演算步骤.

15. (本题满分 10 分)

求曲线 $y = \frac{x^{1+x}}{(1+x)^x}$ ($x > 0$) 的斜渐近线方程.

解析: $\lim_{x \rightarrow +\infty} \frac{y}{x} = \lim_{x \rightarrow +\infty} \frac{x^{1+x}}{(1+x)^x x}$

$$= \lim_{x \rightarrow +\infty} \frac{x^x}{(1+x)^x}$$

$$= \lim_{x \rightarrow +\infty} \frac{e^{x \ln x}}{e^{x \ln(1+x)}}$$

$$= \lim_{x \rightarrow +\infty} e^{x(\ln x - \ln(1+x))}$$

$$= \lim_{x \rightarrow +\infty} e^{x \ln \frac{x+1}{1+x}}$$

$$= \lim_{x \rightarrow +\infty} e^{x \ln \left(1 - \frac{1}{1+x}\right)}$$

$$= \lim_{x \rightarrow +\infty} e^{x \cdot \left(-\frac{1}{1+x}\right)} = e^{-1}$$

$$\lim_{x \rightarrow +\infty} (y - e^{-1}x)$$

$$= \lim_{x \rightarrow +\infty} \left(\frac{x^{1+x}}{(1+x)^x} - e^{-1}x \right)$$

$$= \lim_{x \rightarrow +\infty} x \left(\frac{x^x}{(1+x)^x} - e^{-1} \right)$$

$$= \lim_{x \rightarrow +\infty} x \cdot \left(e^{\frac{x \ln x}{1+x}} - e^{-1} \right)$$

$$= \lim_{x \rightarrow +\infty} x e^{-1} \left(e^{\frac{x \ln x}{1+x} + 1} - 1 \right)$$

$$= \lim_{x \rightarrow +\infty} e^{-1} x \cdot \left(x \ln \frac{x}{1+x} + 1 \right)$$

$$= \lim_{t \rightarrow 0^+} e^{-1} \frac{\frac{1}{t} \cdot \ln \frac{\frac{1}{t}}{1 + \frac{1}{t}} + 1}{t}$$

$$= \lim_{t \rightarrow 0^+} e^{-1} \frac{\ln \frac{1}{t+1} + t}{t^2}$$

$$= \lim_{t \rightarrow 0^+} e^{-1} \frac{t - \ln(1+t)}{t^2} = \frac{1}{2} e^{-1}$$

∴ 曲线的斜渐近线方程为 $y = e^{-1}x + \frac{1}{2}e^{-1}$

16. (本题满分 10 分)

已知函数 $f(x)$ 连续且 $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$, $g(x) = \int_0^1 f(xt) dt$, 求 $g'(x)$ 并证明 $g'(x)$ 在 $x=0$ 处连续.

解析：因为 $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$ $\therefore f(0) = \lim_{x \rightarrow 0} f(x) = 0$

所以 $g(0) = \int_0^1 f(0) dt = 0$

因为 $g(x) = \int_0^1 f(xt) dt \stackrel{xt=u}{=} \frac{1}{x} \int_0^x f(u) du$

当 $x \neq 0$ 时， $g'(x) = \frac{xf(x) - \int_0^x f(u) du}{x^2}$

当 $x = 0$ 时， $g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\int_0^x f(u) du}{x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{f(x)}{x} = \frac{1}{2}$

$$\therefore g'(x) = \begin{cases} \frac{\int_0^x f(u) du}{x^2}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$

又因为 $\lim_{x \rightarrow 0} g'(x) = \lim_{x \rightarrow 0} \frac{xf(x) - \int_0^x f(u) du}{x^2}$

$$= \lim_{x \rightarrow 0} \left[\frac{f(x)}{x} - \frac{\int_0^x f(u) du}{x^2} \right] = 1 - \frac{1}{2} = \frac{1}{2}$$

$\therefore g'(x)$ 在 $x = 0$ 处连续

17. (本题满分 10 分)

求二元函数 $f(x, y) = x^3 + 8y^3 - xy$ 的极值

解析：求一阶导可得

$$\frac{\partial f}{\partial x} = 3x^2 - y$$

$$\frac{\partial f}{\partial y} = 24y^2 - x$$

$$\text{令 } \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \text{ 可得 } \begin{cases} x = 0 \\ y = 0 \end{cases} \begin{cases} x = \frac{1}{6} \\ y = \frac{1}{12} \end{cases}$$

求二阶导可得

$$\frac{\partial^2 f}{\partial x^2} = 6x \quad \frac{\partial^2 f}{\partial x^2 y} = -1 \quad \frac{\partial^2 f}{\partial y^2} = 48y$$

当 $x = 0, y = 0$ 时. $A = 0. B = -1. C = 0^-$

$AC - B^2 < 0$ 故不是极值.

当 $x = \frac{1}{6}, y = \frac{1}{12}$ 时

$$A = 1, B = -1, C = 4.$$

$AC - B^2 > 0, A = 1 > 0$ 故 $\left(\frac{1}{6}, \frac{1}{12}\right)$ 且极小值

$$\text{极小值 } f\left(\frac{1}{6}, \frac{1}{12}\right) = \left(\frac{1}{6}\right)^3 + 8\left(\frac{1}{12}\right)^3 - 6 \times \frac{1}{12} = -\frac{1}{216}$$

18. 已知 $2f(x) + x^2 f\left(\frac{1}{x}\right) = \frac{x^2 + 2x}{\sqrt{1+x^2}}$, 求 $f(x)$, 并求直线 $y = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$ 与函数 $f(x)$ 所

围图形绕 x 轴旋转一周而成的旋转体的体积。

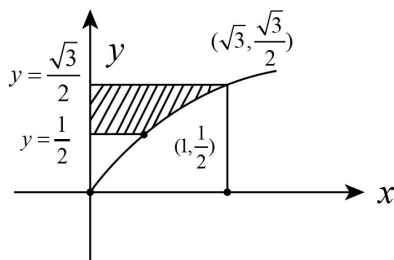
$$\text{解析: } \textcircled{1} \because 2f(x) + x^2 f\left(\frac{1}{x}\right) = \frac{x^2 + 2x}{\sqrt{1+x^2}} \dots \textcircled{1}$$

$$2f\left(\frac{1}{x}\right) + \frac{1}{x^2} f(x) = \frac{\frac{1}{x^2} + 2\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}} = \frac{1+2x}{x\sqrt{1+x^2}} \dots \textcircled{2}$$

$\textcircled{1} \times 2 - \textcircled{2} \times x^2$ 得

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}$$

$\textcircled{2}$



$$\begin{aligned} V &= \pi \cdot \left(\frac{\sqrt{3}}{2}\right)^2 \cdot \sqrt{3} - \pi \left(\frac{1}{2}\right)^2 - \int_1^{\sqrt{3}} \pi \frac{x^2}{x^2 + 1} dx \\ &= \frac{3\sqrt{3}}{4} \pi - \frac{1}{4} \pi - \pi \cdot \sqrt{3} + \frac{\pi^2}{12} \\ &= \frac{\pi^2}{12} - \frac{1}{4} \pi - \frac{\sqrt{3}}{4} \pi \end{aligned}$$

19. (本题满分 10 分)

平面 D 由直线 $x=1, x=2, y=x$ 与 x 轴围成, 计算 $\iint_D \frac{\sqrt{x^2+y^2}}{x} dx dy$.

解析: 积分区域如图:

$$\iint_D \frac{\sqrt{x^2+y^2}}{x} dx dy$$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} \frac{r}{r \cos\theta} \cdot r dr$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} \cdot \frac{1}{2} r^2 \Big|_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} \cdot \frac{3}{\cos^2\theta} d\theta$$

$$= \frac{3}{2} \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta = \frac{3}{2} \int_0^{\frac{\pi}{4}} \sec \theta d \tan \theta$$

$$= \frac{3}{2} \left[\sec \theta \tan \theta \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2 \sec \theta d\theta \right]$$

$$= \frac{3}{2} \left[\sqrt{2} - \int_0^{\frac{\pi}{4}} (\sec^2 \theta - 1) \sec \theta d\theta \right]$$

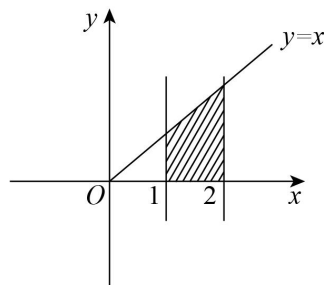
$$= \frac{3}{2} \left(\sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta + \int_0^{\frac{\pi}{4}} \sec \theta d\theta \right)$$

$$= \frac{3}{2} \left(\sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta + \ln |\sec \theta + \tan \theta| \Big|_0^{\frac{\pi}{4}} \right)$$

$$= \frac{3}{2} \left(\sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta + \ln(\sqrt{2}+1) \right)$$

所以 $\int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta = \frac{\sqrt{2}}{2} + \frac{1}{2} \ln(\sqrt{2}+1)$

$$\iint_D \frac{\sqrt{x^2+y^2}}{x} dx dy = \frac{3}{2} \left(\frac{\sqrt{2}}{2} + \frac{1}{2} \ln(\sqrt{2}+1) \right)$$



$$= \frac{3}{4} [\sqrt{2} + \ln(\sqrt{2} + 1)]$$

20. (本题满分 11 分)

设函数 $f(x) = \int_1^x e^{t^2} dt$.

证: 存在 $\xi \in (1, 2)$, $f(\xi) = (2 - \xi)e^{\xi^2}$;

(2) 证: 存在 $\eta \in (1, 2)$, $f(2) = \ln 2 \cdot \eta e^{\eta^2}$.

证明 (1) 构造辅助函数 $F(x) = f(x)(x-2) = (x-2) \int_1^x e^{t^2} dt$

显然 $F(1) = 0, F(2) = 0$, 又 $F(x)$ 在 $[1, 2]$ 连续, $(1, 2)$ 上可导,

由罗尔定理知 $\exists \xi \in (1, 2)$, 使得 $F'(\xi) = 0$

又因为 $F'(x) = \int_1^x e^{t^2} dt + (x-2)e^{x^2} = f(x) + (x-2)e^{x^2}$

所以 $f(\xi) = (2 - \xi)e^{\xi^2}$.

令 $g(x) = \ln x$ 由柯西中值定理得 $\exists \eta \in (1, 2)$

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f(2)}{\ln 2} = \frac{e^{\eta^2}}{\frac{1}{\eta}} = \eta e^{\eta^2}$$

使得

即 $f(2) = \ln 2 \cdot \eta e^{\eta^2}$

21. (本题满分 11 分)

设曲线 $y = f(x)$ 可导, 且 $f'(x) > 0 (x \geq 0)$, $f(x)$ 的图象过原点 O

曲线上任意一点 M 的切线与 X 轴交于 T, $MP \perp x$ 轴, 曲线 $y = f(x)$, MP , x 轴围成的面积与 ΔMTP 面积比为 3: 2, 求曲线方程.

解析: 设切点 M 坐标为 (x, y) , 则过 M 的切线方程为

$$Y - y = y'(X - x)$$

令 $Y = 0$ 得 $X = x - \frac{y}{y'}$

由题意得

$$\frac{\int_0^x f(t)dt}{\frac{1}{2} \cdot \frac{y}{y'} \cdot y} = \frac{3}{2}$$

整理并求导得 $3yy'' - 2y'^2 = 0$

令 $y' = p$ $y'' = p \frac{dp}{dy}$ 代入上式得

$$3yp \frac{dp}{dy} - 2p^2 = 0$$

解得 $p = C_1 y^{\frac{2}{3}}$

即 $y' = C_1 y^{\frac{2}{3}}$

$$\frac{dy}{y^{\frac{2}{3}}} = C_1 dx$$

$$3y^{\frac{1}{3}} = C_1 x + C_2 \quad \text{由 } y(0) = 0 \text{ 得 } C_2 = 0.$$

$$3y^{\frac{1}{3}} = C_1 x$$

$$y = Cx^3$$

22. (本题满分 11 分)

设二次型 $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2ax_1x_2 + 2ax_1x_3 + 2ax_2x_3$ 经可逆线性变换

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ 得 } g(y_1, y_2, y_3) = y_1^2 + y_2^2 + 4y_3^2 + 2y_1y_2.$$

(1) 求 a 的值;

(2) 求可逆矩阵 P .

解析:

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}$$

(1) 令 $f(x_1, x_2, x_3)$ 的矩阵

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$f(y_1, y_2, y_3)$ 的矩阵

A 与 B 合同, 则 $r(A) = r(B)$.

由于 $|B| = 0$, 故 $r(B) < 3$, 故 $|A| = 0$.

$$|A| = \begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix} = (2a+1)(a-1)^2 = 0$$

而

$$\text{解得 } a = -\frac{1}{2} \text{ 或 } a = 1.$$

当 $a = 1$ 时, $r(A) = 1$, 而 $r(B) = 2$. 故舍去

$$\text{所以 } a = -\frac{1}{2}.$$

(2) 当 $a = -\frac{1}{2}$ 时, 利用配方法把 $f(x_1, x_2, x_3)$ 化为规范形.

$$\begin{aligned} f(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3 \\ &= \left(x_1 - \frac{1}{2}x_2 - \frac{x_3}{2}\right)^2 + \frac{3}{4}x_2^2 + \frac{3}{4}x_3^2 - \frac{3}{2}x_2x_3 \\ &= \left(x_1 - \frac{1}{2}x_2 - \frac{x_3}{2}\right)^2 + \frac{3}{4}(x_2 - x_3)^2 \end{aligned}$$

$$\text{令 } \begin{cases} z_1 = x_1 - \frac{1}{2}x_2 - \frac{x_3}{2} \\ z_2 = \frac{\sqrt{3}}{2}(x_2 - x_3) \\ z_3 = x_3 \end{cases} \quad \text{即令 } P_1 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$Z = P_1X, \text{ 则 } f(x_1, x_2, x_3) = z_1^2 + z_2^2$$

利用配方法把 $f(y_1, y_2, y_3)$ 化为规范形.

$$f(y_1, y_2, y_3) = y_1^2 + y_2^2 + 2y_1y_2 + 4y_3^2 = (y_1 + y_2)^2 + 4y_3^2$$

$$\text{令 } \begin{cases} z_1 = y_1 + y_2 \\ z_2 = 2y_3 \\ z_3 = y_2 \end{cases} \quad \text{即令 } P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \quad Z = P_2 Y.$$

$$\text{则 } f(y_1, y_2, y_3) = z_1^2 + z_2^2.$$

$$\text{故 } P_1 X = P_2 Y \quad \text{即 } X = P_1^{-1} P_2 Y.$$

$$\text{所以 } P = P_1^{-1} P_2.$$

$$P_1^{-1} = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 1 \\ 0 & \frac{2}{\sqrt{3}} & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

由于

$$\text{故 } P = P_1^{-1} P_2 = \begin{bmatrix} 1 & 2 & \frac{2}{3}\sqrt{3} \\ 0 & 1 & \frac{4}{3}\sqrt{3} \\ 0 & 1 & 0 \end{bmatrix}$$

23. (本题满分 11 分)

设 A 为 2 阶矩阵, $P = (\alpha, A\alpha)$, 其中 α 是非零向量且不是 A 的特征向量.

(1) 证明 P 为可逆矩阵.

(2) 若 $A^2\alpha + A\alpha - 6\alpha = 0$, 求 $P^{-1}AP$, 并判断 A 是否相似于对角矩阵.

解析:

(1) $\alpha \neq 0$ 且 $A\alpha \neq \lambda\alpha$.

故 α 与 $A\alpha$ 线性无关.

$$\text{则 } r(\alpha, A\alpha) = 2$$

则 P 可逆.

(2) 法一: 由已知有 $A^2\alpha = -A\alpha + 6\alpha$

于是 $AP = A(\alpha, A\alpha) = (A\alpha, A^2\alpha) = (A\alpha, -A\alpha + 6\alpha)$

$= (\alpha, A\alpha) \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$, 故有 $P^{-1}AP = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$, $\therefore P$ 可逆

\therefore 可得 A 与 $\begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$ 相似, 又 $\begin{vmatrix} \lambda & -6 \\ -1 & \lambda+1 \end{vmatrix} = (\lambda+3) \cdot (\lambda-2) = 0$

$\Rightarrow \lambda_1 = -3, \lambda_2 = 2$

\therefore 可得 A 的特征值也为 $-3, 2$ 于是 A 可相似对角化

方法二 $P^{-1}AP$ 同方法一

由 $A^2\alpha + A\alpha - 6\alpha = 0$

下面是证明 A 可相似对角化

$$(A^2 + A - 6E)\alpha = 0$$

设 $(A+3E)(A-2E)\alpha = 0$

由 $\alpha \neq 0$ 得 $(A^2 + A - 6E)x = 0$ 有非零解

故 $|(A+3E)(A-2E)| = 0$

得 $|A+3E| = 0$ 或 $|A-2E| = 0$

若 $|A+3E| \neq 0$ 则有 $(A-2E)\alpha = 0$ 故 $A\alpha = 2\alpha$ 与题意矛盾

故 $|A+3E| = 0$ 同理可得 $|A-2E| = 0$

于是 A 的特征值为 $\lambda_1 = -3, \lambda_2 = 2$.

A 有 2 个不同特征值故 A 相似对角化
